Product of groups

Sasha Patotski

Cornell University

ap744@cornell.edu

November 29, 2015

Last times

Definition

If G, H are groups, a map $\varphi\colon G \to H$ is called a **homomorphism** if

$$\varphi(a * b) = \varphi(a) * \varphi(b)$$

For a homomorphism φ, necessarily $\varphi(e_G) = e_H$ and $\varphi(a^{-1}) = \varphi(a)^{-1}$.

Definition

The **kernel** of a homomorphism φ is $\ker \varphi = \{a \in G \mid \varphi(a) = e\}$
The **image** of a homomorphism φ is $\operatorname{im} \varphi = \{\varphi(a) \in H \mid a \in G\}$

Definition

Two groups G, H are **isomorphic** if there is a homomorphism $\varphi\colon G \to H$ which is bijective. Such φ is called an **isomorphism**.

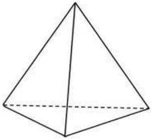
Examples

- Between **any** groups G, H there is a **trivial** homomorphism $\varphi\colon G \to H$, given by $\varphi(g) = e_H$, for all $g \in G$.
- There are no **nontrivial** homomorphisms $\mathbb{Z}/m \to \mathbb{Z}$.
- For any **abelian** group G, the map $\varphi_m\colon G \to G$ given by $g \mapsto g^m$ is a homomorphism.
- The same map for **non-abelian group** is **not necessarily** a homomorphism.
- The group of symmetries of an equilateral triangle is isomorphic to S_3.
- The group of orientation preserving symmetries of a regular n-gon is isomorphic to $\mathbb{Z}/n$.

Tetrahedron

Theorem

The group G of symmetries of a regular tetrahedron is isomorphic to S_4.



- There is an obvious homomorphism $\varphi\colon G \to S_4$, sending a symmetry to corresponding permutation of vertices.
- It is injective: if a symmetry fixes all vertices, it must be the identity symmetry.
- Every transposition of neighbors is in the image $\mathrm{im}(\varphi)$.
- But $\mathrm{im}(\varphi)$ is a subgroup.
- Since S_4 is generated by these transpositions, $\mathrm{im}(\varphi) = S_4$. Done.

Product of groups

Definition

Let G, H be two groups. Define $G \times H$ to be the set of all pairs (g, h) with $g \in G, h \in H$. Define an operation $*$ on it by

$$(g, h) * (g', h') := (gg', hh')$$

Product of groups

Definition

Let G, H be two groups. Define $G \times H$ to be the set of all pairs (g, h) with $g \in G, h \in H$. Define an operation $*$ on it by

$$(g, h) * (g', h') := (gg', hh')$$

- Prove that $(G \times H, *)$ is again a group.

Product of groups

Definition

Let G, H be two groups. Define $G \times H$ to be the set of all pairs (g, h) with $g \in G, h \in H$. Define an operation $*$ on it by

$$(g, h) * (g', h') := (gg', hh')$$

- Prove that $(G \times H, *)$ is again a group.
- It's called the **product of groups** G, H.

Definition

Let G, H be two groups. Define $G \times H$ to be the set of all pairs (g, h) with $g \in G, h \in H$. Define an operation $*$ on it by

$$(g, h) * (g', h') := (gg', hh')$$

- Prove that $(G \times H, *)$ is again a group.
- It's called the **product of groups** G, H.
- Prove that the subsets $G \times \{e\} = \{(g, e) \mid g \in G\}$ and $\{e\} \times H$ are two subgroups of $G \times H$, isomorphic to G and H respectively.

Product of groups

Definition

Let G, H be two groups. Define $G \times H$ to be the set of all pairs (g, h) with $g \in G, h \in H$. Define an operation $*$ on it by

$$(g, h) * (g', h') := (gg', hh')$$

- Prove that $(G \times H, *)$ is again a group.
- It's called the **product of groups** G, H.
- Prove that the subsets $G \times \{e\} = \{(g, e) \mid g \in G\}$ and $\{e\} \times H$ are two subgroups of $G \times H$, isomorphic to G and H respectively.
- Prove that the two subgroups commute with each other.

Product of groups

Definition

Let G, H be two groups. Define $G \times H$ to be the set of all pairs (g, h) with $g \in G, h \in H$. Define an operation $*$ on it by

$$(g, h) * (g', h') := (gg', hh')$$

- Prove that $(G \times H, *)$ is again a group.
- It's called the **product of groups** G, H.
- Prove that the subsets $G \times \{e\} = \{(g, e) \mid g \in G\}$ and $\{e\} \times H$ are two subgroups of $G \times H$, isomorphic to G and H respectively.
- Prove that the two subgroups commute with each other.
- The natural projections $G \times H \to G$ and $G \times H \to H$ are homomorphisms of groups.

Product of groups

Definition

Let G, H be two groups. Define $G \times H$ to be the set of all pairs (g, h) with $g \in G, h \in H$. Define an operation $*$ on it by

$$(g, h) * (g', h') := (gg', hh')$$

- Prove that $(G \times H, *)$ is again a group.
- It's called the **product of groups** G, H.
- Prove that the subsets $G \times \{e\} = \{(g, e) \mid g \in G\}$ and $\{e\} \times H$ are two subgroups of $G \times H$, isomorphic to G and H respectively.
- Prove that the two subgroups commute with each other.
- The natural projections $G \times H \to G$ and $G \times H \to H$ are homomorphisms of groups.
- The product of **abelian** groups is a again abelian.

Examples

- S_3 is not isomorphic to $A_3 \times \mathbb{Z}/2$ (can replace 3 by n).

Examples

- S_3 is not isomorphic to $A_3 \times \mathbb{Z}/2$ (can replace 3 by n).

Examples

- S_3 is not isomorphic to $A_3 \times \mathbb{Z}/2$ (can replace 3 by n).
- For odd n, the group $GL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^\times \times SL_n(\mathbb{R})$.

Examples

- S_3 is not isomorphic to $A_3 \times \mathbb{Z}/2$ (can replace 3 by n).
- For odd n, the group $GL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^\times \times SL_n(\mathbb{R})$.
- $\mathbb{Z}/4$ is not isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Examples

- S_3 is not isomorphic to $A_3 \times \mathbb{Z}/2$ (can replace 3 by n).
- For odd n, the group $GL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^\times \times SL_n(\mathbb{R})$.
- $\mathbb{Z}/4$ is not isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.
- $\mathbb{Z}/6$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/3$ (can replace $2, 3, 6$ by m, n, mn with relatively prime m, n).

Product of groups

Theorem

Let G be a group, and $H, K \subseteq G$ be two subgroups satisfying the following three properties:

1. $H \cap K = \{e\}$;
2. *elements of H commute with elements of K, i.e. $hk = kh$ for any $h \in H, k \in K$;*
3. $G = HK$*, i.e. any g can be represented as $g = hk$ with $h \in H, k \in K$.*

Then $G \simeq H \times K$.

Proof.

□

Product of groups

Theorem

Let G be a group, and $H, K \subseteq G$ be two subgroups satisfying the following three properties:

1. $H \cap K = \{e\}$;
2. *elements of H commute with elements of K, i.e. $hk = kh$ for any $h \in H, k \in K$;*
3. $G = HK$, *i.e. any g can be represented as $g = hk$ with $h \in H, k \in K$.*

Then $G \simeq H \times K$.

Proof.

- Consider $p \colon H \times K \to G$ defined by $p(h, k) = hk \in G$. This is a homomorphism.

$\square$

Product of groups

Theorem

Let G be a group, and $H, K \subseteq G$ be two subgroups satisfying the following three properties:

1. $H \cap K = \{e\}$;
2. *elements of H commute with elements of K, i.e. $hk = kh$ for any $h \in H, k \in K$;*
3. $G = HK$, *i.e. any g can be represented as $g = hk$ with $h \in H, k \in K$.*

Then $G \simeq H \times K$.

Proof.

- Consider $p\colon H \times K \to G$ defined by $p(h, k) = hk \in G$. This is a homomorphism.
- It is injective and surjective, and hence an isomorphism.

$\square$

Symmetries of a cube

Theorem

The group G of symmetries of a cube is isomorphic to $S_4 \times \mathbb{Z}/2$.